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Derivation of the Transport Equation for Electrons Moving Through Random Impurities

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A single (nonrelativistic, spinless) electron subject to a constant external electric field interacts with impurities located on an infinitely extended lattice by a potential of random strength. The random strength is given by a field of Gaussian random variables. We show the existence of the averaged dynamics and prove that in the weak coupling limit, $\lambda \to 0$, $\lambda^2 t = \tau$ fixed, one obtains the usual transport equation for the velocity distribution.

KEY WORDS: Transport equation; random impurities; electrical conduction, van Hove limit.

1. INTRODUCTION

One of the central problems of nonequilibrium statistical mechanics is the rigorous derivation of macroscopic evolution equations from the microscopic dynamics. In the present paper we investigate this problem for the motion of electrons through random impurities. This is not an artificial case, since for a real solid at sufficiently low temperatures impurity scattering will be the dominant source of collisions for conduction electrons. The scattering by thermal vibrations of the crystal and by other conduction electrons becomes increasingly weak as the temperature drops. As a consequence, the physics of impurity scattering has been studied extensively (see Ref. 1 and references therein).

We consider here a single (nonrelativistic, spinless) electron interacting with impurities by a smooth potential $V(r - x)$, where r is the position of the electron and x is the position of the impurity. One is tempted to express the random character of the impurities by assuming their location to be random. However, a more tractable model results if we assume the impurities to be

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located on a fixed, infinite lattice and the interaction potential to be of random strength, which means that the potential is of the form $v_x V(r - x)$, where v_x is a real random variable. We will be interested in a situation where the coupling between impurities and electron is weak and therefore scale the interaction strength by $\lambda > 0$. Finally, we place the whole solid in a constant external electric field E.

The total microscopic Hamiltonian then reads

$$H^\lambda = -\Delta + \lambda^2 E \cdot r + \lambda \sum_{x \in \Gamma} v_x V(r - x) \tag{1}$$

where Γ is a lattice, and the time evolution of a density matrix ρ is given by the Liouville–von Neumann equation

$$(d/dt)\rho = -i[H^\lambda, \rho] \tag{2}$$

The reason for scaling the electric field by λ^2 can be easily understood in terms of a ball moving through a viscous fluid. If the viscosity is decreased, then in order to maintain a stable situation, we should also decrease the applied force. It turns out that in our model the "friction" exerted by the random potential on the electron decreases proportional to λ^2 as $\lambda \to 0$ — hence our scaling.

The precise nature of the random field $\{v_x | x \in \Gamma\}$ should be rather immaterial for small λ. However, for mathematical reasons, we assume that $\{v_x | x \in \Gamma\}$ is a field of Gaussian random variables with mean zero and covariance $\langle v_x v_y \rangle$ depending only on the relative distance of the sites x and y. Since the random variables v_x are unbounded, H^λ is only a formal Hamiltonian. We can easily arrange the v_x in such a way that the electron is already at infinity in a finite time, which means that the dynamics is not uniquely defined for such a value of the v_x. We will have to show that such values of the v_x are of probability zero.

Given the hint that the coupling λ is small, one applies second-order perturbation theory (the first order vanishes since $\langle v_x \rangle = 0$) to derive *formally* the transport equation for the velocity distribution $\rho(k)$

$$(d/dt)\rho(k) + E \cdot \nabla \rho(k) = \int dk' \, K(k, k')[\rho(k') - \rho(k)] \tag{3}$$

The collision kernel K is given in terms of the Fourier transform $\hat{V}$ of the potential V and of the Fourier transform $\hat{g}$ of the covariance $\langle v_x v_0 \rangle$ as

$$K(k, k') = \delta(k^2 - k'^2) |\hat{V}(k - k')|^2 \hat{g}(k - k') \tag{4}$$

The aim of the present paper is to show how one can rigorously derive the transport equation (3) from the microscopic equations of motion (2). We want to emphasize that by the same method we can also treat the derivation of the transport equation for the truly dynamical model, where at every lattice site there sits a harmonic oscillator coupled to the electron by a one-phonon–electron interaction and not coupled to all other oscillators (cf. Section 4).

Twenty years ago van Hove[2] investigated large systems with a Hamiltonian roughly of the form (1). He used a perturbation expansion in λ (Dyson expansion) and argued that the terms of order $2n$ should behave as $\lambda^{2n}t^n$. (This should be contrasted with the "crude" estimate giving an order of $\lambda^{2n}t^{2n}$.) This indicates that the proper limit to look at is

$$\lambda \to 0, \qquad \lambda^2 t = \tau \text{ fixed} \tag{5}$$

which is the van Hove long-time, weak coupling limit. Furthermore, van Hove made plausible that most terms of order $2n$ should even behave as $\lambda^{2n}t^{n-\epsilon}$, $\epsilon > 0$, and therefore vanish in the weak coupling limit. The remaining terms should add up in such a way as to result in the transport equation (3). Amazingly enough, van Hove is right.

At first sight, the weak coupling limit (5) might look surprising. It should be considered as a device to isolate a, for small λ, dominant exponential time decay from "background" contributions. To illustrate this point, two examples are in order. The first one is the so-called Friedrichs or Lee model, which is so simple that the weak coupling limit can be followed analytically. The unperturbed Hamiltonian H_0 has a single eigenvalue embedded in the continuous spectrum. The perturbation λV couples this eigenvalue to the continuum. One is interested in the survival probability of the eigenstate. For small λ one has typically three time scales: an extremely short initial time period followed by a long time period with almost pure exponential decay and, finally, a long-time tail with a power law decay. On the (reduced) τ time scale the exponential decay rate stays practically constant and in the limit as $\lambda \to 0$ the pure exponential decay survives.

The other example is somewhat more closely related to the present model. Bruin[3] made a molecular dynamics study of the two-dimensional Lorentz model, where a hard disk moves through 2000 random point scatterers. He computed the velocity–velocity autocorrelation function for small scatterer densities ρ. On the reduced time scale $\tau = t\rho$ one finds the dominant exponential decay $\exp(-\frac{4}{3}\tau)$ as given by the Boltzmann equation and some small contributions which vanish in the limit $\rho \to 0$, $t \to \infty$, $t\rho = \tau$ fixed. This example shows also, if one believes in the analogy, that the diffusion constant and the conductivity as computed from the transport equation (3) give the coefficient of the lowest order term (proportional to λ^{-2}) in a λ expansion (which is very likely not a power series expansion) of these transport coefficients.

On a more intuitive level the weak coupling limit means that for small λ in a unit τ time interval the electron interacts weakly with many impurities. Only in this limit can we hope to approximate the averaged microscopic time evolution (which contains complicated memory effects) by as simple a Markovian time evolution as (3).

Martin and Emch[4] studied rigorously the van Hove weak coupling limit for a model which is a simplified version of the present one. They use a momentum cutoff, which means that the electron lives on the lattice of integers $\mathbb{Z}^3$ (the momentum space is the first Brioullin zone $[-\pi, \pi]^3$). They assume a δ-type interaction potential and set $E = 0$. They studied the averaged matrix elements $(\langle\psi|\exp(iH^0 t)\exp(-iH^\lambda t)\psi\rangle)_{\rm av}$ in the weak coupling limit and did not derive the transport equation (3). The importance of the work by Martin and Emch lies, besides its originating the model, in the fact that they revealed the mechanism responsible for the existence of the weak coupling limit: The spreading of the quantum mechanical wave packet allows one to control the perturbation expansion in such a way that in the limit $\lambda \to 0$, $\lambda^2 t = \tau$ fixed, the transport equation results.

The paper is organized in the following way: We choose a Hamiltonian $H_\Lambda{}^\lambda$ that is of the form (1) with the sum only over the finitely many sites in $\Lambda \subset \Gamma$. Then the existence of the averaged dynamics $T_{t,\Lambda}^\lambda$ for observables in the interaction picture can easily be established. In Section 2 we show the existence of the averaged dynamics in the infinite-volume limit

$$\lim_{\Lambda\to\Gamma} T_{t,\Lambda}^\lambda A = T_t^\lambda A \tag{6}$$

where A is a function of the momentum. In Section 3 we establish the existence of the weak coupling limit

$$\lim_{\lambda\to 0} T_{\lambda^{-2}\tau}^\lambda A \tag{7}$$

and show that this limit is given by the integrated form of (3). The transport equation can be derived under the following conditions: (i) a sufficiently rapidly decreasing potential V and covariance $\langle v_x v_0\rangle$; (ii) a spatial dimension of three or greater; and (iii) $0 \leqslant \tau < \tau_0$, where τ_0 is roughly inversely proportional to the sum $\sum_{x\in\Gamma} |\langle v_x v_0\rangle|$ and depends in a more complicated way on the potential V [cf. (54)].

The restriction (ii) comes from the fact that the spreading of the quantum mechanical wave packet, used in an essential way in the estimates, is proportional to $|t|^{-\nu/2}$, which is integrable only for $\nu \geqslant 3$. Condition (iii) reflects the use of a perturbation expansion whose convergence on the τ time scale can be controlled only up to τ_0.

2. EXISTENCE OF THE AVERAGED DYNAMICS IN THE INFINITE-VOLUME LIMIT

The motion of the free electron in ν-dimensional space under the influence of an electric field $\lambda^2 E$ is described by the Hamiltonian

$$H_0{}^\lambda = -\Delta + \lambda^2 E\cdot r \tag{8}$$

acting in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^\nu)$. The Hamiltonian $H_0{}^\lambda$ defined on smooth functions has a unique self-adjoint extension. The unitary group generated by $H_0{}^\lambda$ is denoted by $U_0{}^\lambda(t) = \exp(-iH_0{}^\lambda t)$. We assume that the random impurities are located on a lattice $\Gamma \subset R^\nu$ interacting with the electron by a smooth, rapidly decreasing potential V:

$$H_\Lambda{}^\lambda = -\Delta + \lambda^2 E\cdot r + \lambda \sum_{x\in\Lambda\subset\Gamma} v_x V(r - x) \tag{9}$$

where $v_x \in \mathbb{R}$ and $V(\cdot)$ acts as multiplication operator. (Later, we will allow potentials with somewhat less restrictive decay properties at infinity.) If Λ is a finite subset of Γ, then $H_\Lambda{}^\lambda$ defined on smooth functions has a unique self-adjoint extension. We denote by $U_\Lambda{}^\lambda(t)$ the unitary group generated by $H_\Lambda{}^\lambda$.

To express the random character of the impurities we let $\{v_x | x \in \Gamma\}$ be a translation-invariant, mean-zero field of Gaussian random variables, not necessarily independent. The field is uniquely characterized by its covariance

$$\langle v_x v_y \rangle = g_{x-y}, \qquad x, y \in \Gamma \tag{10}$$

where

$$g \in l_1(\Gamma), \qquad \hat{g}(k) = (\mathrm{BZ})^{-1/2} \sum_{x\in\Gamma} e^{-ixk} g_x > 0 \tag{11}$$

for all $k \in \mathrm{BZ}(\Gamma)$, the first Brillouin zone of Γ. BZ is the volume of the first Brillouin zone. Any odd correlation function vanishes and the even correlation functions are given by the rule

$$\langle v_{x_1} \cdots v_{x_{2n}} \rangle = \sum_p \prod_{j=1}^{n} g_{x_{p(2j-1)} - x_{p(2j)}} \tag{12}$$

$x_1,\dots, x_{2n} \in \Gamma$, where the sum runs over all $(2n)!/(n!\, 2^n)$ pairings of the integers $\{1,\dots, 2n\}$. [This is the subset of all permutations of $\{1,\dots, 2n\}$ such that $p(2j-1) < p(2j)$ and $p(2j-1) < p(2j+1)$.] For finite Λ we use the projection $\langle\cdot\rangle_\Lambda$ of the Gaussian measure to the cylinder set $\mathbb{R}^{|\Lambda|}$.

In the following we will be interested in the time evolution of observables $A \in \mathscr{A} \subset B(\mathscr{H})$, which are functions of the momentum, i.e., we will study only the time evolution of the velocity distribution. Whereas the present section certainly generalizes, it is not clear whether the analysis of the weak coupling limit can be extended beyond $\mathscr{A}$. It should be noted that $\mathscr{A}$ is neither invariant under $U_\Lambda{}^\lambda(t)$ nor invariant under the averaged time evolution. To define the averaged dynamics for a finite $\Lambda \subset \Gamma$ (for observable $A \in \mathscr{A}$ in the interaction picture) we expand in terms of a Dyson series:

$$\begin{aligned}(\phi, T_{t,\Lambda}^\lambda A\psi) &\equiv \langle(\phi, U_0{}^\lambda(t) U_\Lambda{}^\lambda(-t) A U_\Lambda{}^\lambda(t) U_0{}^\lambda(-t)\psi)\rangle_\Lambda \\ &= \sum_{n=0}^{\infty} (-i\lambda)^n \int_{0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant t} dt_n \cdots dt_1 \sum_{x_1,\dots,x_n\in\Lambda} \langle v_{x_n} \cdots v_{x_1}\rangle_\Lambda \\ &\quad \times (\phi, [V(r - x_n, t_n), [\dots, [V(r - x_1, t_1), A]\dots]\psi)\end{aligned} \tag{13}$$

where $V(\cdot, t) = U_0^\lambda(-t) V U_0^\lambda(t)$. We have the bound

$$|(\phi, U_0^\lambda(t) U_\Lambda^\lambda(-t) A U_\Lambda^\lambda(t) U_0^\lambda(-t)\psi)| \leqslant \|\phi\| \|\psi\| \|A\| e^{2\lambda t c} \tag{14}$$

where

$$c = \left(\sup_r \sum_{x \in \Lambda} |V(r - x)|\right)\left(\sup_{y \in \Lambda} |v_y|\right)$$

The right-hand side of (14) is integrable by the assumption on g. Therefore, the averaged dynamics of the electron interacting with finitely many impurities exists.

To describe the motion of the electron in the infinitely extended field of random impurities, it is natural to take the limit of $T_{t,\Lambda}^\lambda$ as Λ tends to Γ. We will show that this limit exists for a suitable class of observables $A \in \mathscr{A}$.

Before we state the theorem we note that $U_0^\lambda(t)$ has a very simple representation in Fourier space. Let f: $\mathbb{R}^\nu \to \mathbb{R}$ be the solution of the differential equation

$$\lambda^2 E \cdot \nabla f(k) = k^2 \tag{15}$$

and let $V_E(t)$ be the shift by $\lambda^2 Et$: $(V_E(t)\hat{\psi})(k) = \hat{\psi}(k + \lambda^2 Et)$. Then

$$\widehat{U_0^\lambda(t)\psi} = e^{if} V_E(t) e^{-if} \hat{\psi} \tag{16}$$

Theorem 1. Let $V \in \mathscr{S}(\mathbb{R}^\nu)$ and $A \in B(\mathscr{H})$ be a continuous function of the momentum. Then in the strong topology of $B(\mathscr{H})$

$$\lim_{\Lambda \to \Gamma} T_{t,\Lambda}^\lambda A = T_t^\lambda A \tag{17}$$

exists. T_t^λ is defined through the norm convergent series

$$T_t^\lambda A = \sum_{n=0}^\infty (i\lambda)^{2n} \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_{2n} \leqslant t} dt_{2n} \cdots dt_1$$

$$\times \sum_{x_1, \ldots, x_{2n} \in \Gamma} \langle v_{x_{2n}} \cdots v_{x_1} \rangle [V(r - x_{2n}, t_{2n}), [\ldots, [V(r - x_1, t_1), A] \ldots] \tag{18}$$

Proof. The commutators in (13) introduce permutations of the time arguments that are given as the subset of all permutations of $\{1, \ldots, n\}$ such that for all $k = 1, \ldots, n$

$$\pi(n) > \pi(n-1) > \cdots > \pi(k) = 1 < \pi(k-1) < \cdots < \pi(1) \tag{19}$$

[We will always think of the permutation as ordered from the right, i.e., $\pi(n) \cdots \pi(1)$.] Since by definition any odd correlation of the Gaussian random field vanishes, the sum (13) extends only over even integers. To simplify notation we define

$$R_{n,\Lambda}(t_1,..., t_{2n})A = \sum_{x_1,...,x_{2n}\in\Lambda} \langle v_{x_{2n}} \cdots v_{x_1}\rangle_\Lambda \times [V(r - x_{2n}, t_{2n}), [...[V(r - x_1, t_1), A]...] \quad (20)$$

$$R_{n,\Lambda}(t)A = \int_{0\leqslant t_1\leqslant \cdots \leqslant t_{2n}\leqslant t} dt_{2n} \cdots dt_1\, R_{n,\Lambda}(t_1,..., t_{2n})A$$

Then, using the form of the free evolution in (16), we obtain

$$\begin{aligned}(\phi, R_{n,\Lambda}(t_1,..., t_{2n})A\psi) &= \sum_{\pi,\delta} \mathrm{si}(\pi)(2\pi)^{-\nu n} \sum_{x_1,...,x_{2n}\in\Lambda} \langle v_{x_{2n}} \cdots v_{x_1}\rangle_\Lambda \int_{\mathbb{R}^{\nu(2n+1)}} d\Theta_{2n+1} \cdots d\Theta_1\, \hat{\phi}^*(\Theta_{2n+1}) \\ &\times A(\Theta_{k+\delta})\hat{\psi}(\Theta_1) \exp\{i[f(\Theta_{2n+1}) - f(\Theta_{2n+1} - \lambda^2 E t_{\pi(2n)})]\} \\ &\times \left(\prod_{j=2}^{2n} \exp\{i[f(\Theta_j - \lambda^2 E t_{\pi(j)}) - f(\Theta_j - \lambda^2 E t_{\pi(j-1)})]\}\right) \\ &\times \exp\{i[f(\Theta_1 - \lambda^2 E t_{\pi(1)}) - f(\Theta_1)]\} \\ &\times \left\{\prod_{j=1}^{2n} \hat{V}(\Theta_{j+1} - \Theta_j) \exp[i x_j(\Theta_{j+1} - \Theta_j)]\right\} \end{aligned} \quad (21)$$

where $\delta = 0, 1$ and $\mathrm{si}(\pi) = 1, -1$ depending on the permutation π. The average is bounded by

$$|\langle v_{x_{2n}} \cdots v_{x_1}\rangle|_\Lambda \leqslant [2n!/(n!\, 2^n)](\|g\|_\infty)^n \quad (22)$$

Let $\psi \in \mathscr{S}(\mathbb{R}^\nu)$ and let $A \in C^\infty(\mathbb{R}^\nu)$ as a function of the momentum. Let Δ_j denote the Laplacian acting on functions of Φ_j. We insert in (21)

$$\prod_{j=1}^{2n} (1 - \Delta_j)^{-m}(1 - \Delta_j)^m$$

with some positive integer m. Then

$$\begin{aligned}|(\phi, R_{n,\Lambda}(t_1,..., t_{2n})A\psi)| &\leqslant \frac{2n!\, 2^n}{n!} (\|g\|_\infty)^n (2\pi)^{-\nu n} \int_{\mathbb{R}^{\nu(2n+1)}} d\Theta_{2n+1} \cdots d\Theta_1 \\ &\times \Bigg(\sum_{x_1,...,x_{2n}\in\Lambda} \prod_{j=2}^{2n} |(1 - \Delta_j)^{-m} \\ &\times \exp\{-i[\Theta_j(x_j - x_{j-1}) - \Theta_j^2(t_{\pi(j)} - t_{\pi(j-1)}) + \Theta_j\lambda^2 E(t_{\pi(j)}^2 - t_{\pi(j-1)}^2)]\}| \\ &\times |(1 - \Delta_1)^{-m} \exp[-i(\Theta_1 x_1 - \Theta_1^2 t_{\pi(1)}) + \Theta_1\lambda^2 E t_{\pi(1)}^2]|\Bigg) \\ &\times |\hat{\phi}^*(\Theta_{2n+1})| \left|\prod_{j=1}^{2n} (1 - \Delta_j)^m \hat{V}(\Theta_{j+1} - \Theta_j) A(\Theta_{k+\delta})\hat{\psi}(\Theta_1)\right| \end{aligned} \quad (23)$$

The term in the large parentheses is a $2n$-fold convolution, which we will estimate first. Let G_m be the Fourier transform of $(1 + \lambda^2)^{-m}$. Then

$$\begin{aligned}\sup_{\Theta} \sum_{x\in\Gamma} &|(1 - \Delta)^{-m} \exp[-i(\Theta x - \Theta^2 s + \Theta\lambda^2 Er)]| \\ &= \sup_{\Theta} \sum_{x\in\Gamma} \left| \int dy \{\exp[iy(x - 2\Theta s + \lambda^2 Er)] \exp(iy^2 s)\} G_m(y) \right| \\ &\leqslant \sum_{x\in\Gamma} (1 + x^2)^{-m'} \int dy\, |(1 - \Delta)^{m'} [\exp[(iy^2 s)] G_m(y)]| \end{aligned} \tag{24}$$

where we choose the positive integer m' such that the sum is finite. Expression (24) is bounded by a finite polynomial $C(|s|)$ with positive coefficients of the form

$$C_{\alpha\beta} \int dy\, |y^\alpha D^\beta G_m(y)|$$

where α, β are multiindices and D stands for differentiation. Upon Fourier-transforming, we obtain

$$\begin{aligned}\int dy\, |y^\alpha D^\beta G_m(y)| &= \int dy \left|\int dk\, e^{iky} D^\alpha k^\beta (1 + k^2)^{-m}\right| \\ &\leqslant \int dy\, (1 + y^2)^{-m'} \int dk\, |(1 - \Delta)^{m'} D^\alpha k^\beta (1 + k^2)^{-m}| < \infty \end{aligned} \tag{25}$$

for a suitable choice of m. Therefore we can choose m in such a way that

$$\begin{aligned}&|(\phi, R_{n,\Lambda}(t) A\psi)| \\ &\leqslant \frac{1}{n!} [tC(2t)]^{2n} (\|g\|_\infty)^n \\ &\quad \times \|\phi\| \left\| \int d\Theta_{2n} \cdots d\Theta_1 \prod_{j=1}^{2n} |(1 - \Delta_j)^m \hat{V}(\Theta_{j+1} - \Theta_j) A(\Theta_{k+\delta}) \hat{\psi}(\Theta_1)| \right\| \\ &\leqslant \frac{1}{n!} [tC(t)]^{2n} (\|g\|_\infty)^n (C')^{2n} \|\phi\| \, \|A\|_{\approx} \|\hat{\psi}\|_{\sim} \end{aligned} \tag{26}$$

by our assumption on V, A, and ψ. Here $\|\psi\|_\sim$ is the maximum of the L^1-norms of the derivatives of ψ up to order $2m$ and $\|A\|_\approx$ is the maximum of the sup norms of the derivatives of A up to order $2m$. Condition (26) establishes the strong convergence of (13) for all t uniformly in Λ.

To show that the strong limit

$$\lim_{\Lambda\to\Gamma} R_{n,\Lambda}(t) A\psi$$

exists, we insert in (21)

$$(1 - \Delta_j)^m [1 + (x_j - x_{j-1})^2]^{-m} \tag{27}$$

before the last exponent. Then for $\Lambda \subset \Lambda'$

$$|(\phi, [R_{n,\Lambda'}(t)A - R_{n,\Lambda}(t)A]\psi)| \leqslant \|\phi\| C \sum_{x_1,\ldots,x_{2n}\in\Lambda'\setminus\Lambda} \prod_{j=2}^{2n} [1 + (x_j - x_{j-1})^2]^{-m}(1 + x_1)^{-m} \tag{28}$$

which tends to zero as $\Lambda, \Lambda' \to \Gamma$.

By Lebesgue's dominated convergence theorem, the strong limit

$$\lim_{\Lambda\to\Gamma} T_{t,\Lambda}^{\lambda} A\psi = T_t^{\lambda} A\psi \tag{29}$$

exists and is given by (18) for all $\psi \in \mathscr{S}(\mathbb{R}^\nu)$. Since by definition

$$\|T_{t,\Lambda}^{\lambda} A\| < \|A\| \tag{30}$$

and since $\mathscr{S}(\mathbb{R}^\nu)$ is dense in $\mathscr{H}$, (29) establishes strong convergence of $T_{t,\Lambda}^{\lambda} A$ as $\Lambda \to \Gamma$. Since $C^\infty(\mathbb{R}^\nu)$ is dense in $C(\mathbb{R}^\nu)$ in the sup-norm and because of (30), (29) holds for all $A \in B(\mathscr{H})$ that are a continuous function of the momentum. The norm convergence of (18) will be proved in the next section. ■

The proof shows that our assumption on V was somewhat too stringent. It suffices to choose V such that the derivatives of its Fourier transform up to order $\frac{3}{2}(\nu + 1)$ are integrable. From our later estimates it will follow that, in fact, the series (18) converges in norm if $V, \hat{V} \in L^1(\mathbb{R}^\nu)$.

3. THE WEAK COUPLING LIMIT

Let $C_\infty(\mathbb{R}^\nu)$ be the space of continuous functions on $\mathbb{R}^\nu$ with $f(k) \to 0$ as $|k| \to \infty$. The space $C_\infty(\mathbb{R}^\nu)$ is a Banach space under the sup norm. Let $V \in \mathscr{S}(\mathbb{R}^\nu)$, $g \in s(\Gamma)$. Then the transport equation

$$(d/dt)A - E\cdot\nabla A = L(A) \tag{31}$$

with

$$L(A) = (BZ)^{3/2} \int_{\mathbb{R}^\nu} dk_1\, |\hat{V}(k - k_1)|^2 \hat{g}(k - k_1)\, \delta(k^2 - k_1^2)[-A(k) + A(k_1)] \tag{32}$$

generates a contractive, positivity-preserving semigroup on $C_\infty(\mathbb{R}^\nu)$. [Since we study the time evolution of observables, i.e., of functions of the momentum, the generator in (31) is the dual of the generator in (3).] The Fourier transform $\hat{g}$ is extended periodically out of the first Brillouin zone. We denote by S_t the semigroup generated by (31) and by S_t^0 the semigroup generated by (31) with $L = 0$.

To verify our claim, we note that L is a bounded operator, since

$$\sup_{k\in\mathbb{R}^\nu}\left|\int dk_1\,|\hat{V}(k-k_1)|^2\,\delta(k^2-k_1{}^2)\right|<\infty \tag{33}$$

Let $f\in C_\infty(\mathbb{R}^\nu)$. Then there exists a point $k_0\in\mathbb{R}^\nu$ such that $|f(k_0)| = \|f\|$. If we choose $l_f = f^*(k_0)\,\delta(k-k_0)$ as normalized tangent functional to f, then one verifies directly that L is accretive (Ref. 5, Chap. X.8). Therefore e^{Lt} is a contractive semigroup. Applying the Trotter product formula to e^{Lt} and $S_t{}^0$, we conclude that S_t is a contractive semigroup.

We are now in a position to state our main result.

Theorem 2. Let $V\in\mathscr{S}(\mathbb{R}^\nu)$, $g\in s(\Gamma)$, and $A\in C_\infty(\mathbb{R}^\nu)$. Then the series (19) defining $T_t{}^\lambda$ converges in norm. Let S_t and $S_t{}^0$ be defined as above. If $\nu\geqslant 3$, then there exists a $\tau_0>0$ such that for all $0\leqslant\tau<\tau_0$

$$\lim_{\lambda\to 0} T^\lambda_{\lambda^{-2}\tau}A(p) = (S^0_{-\tau}S_\tau A)(p)\in B(\mathscr{H}) \tag{34}$$

in the weak topology of $B(\mathscr{H})$. Here $A(p)$ denotes the function A of the momentum p as an element of $B(\mathscr{H})$.

Proof. First we establish some notation and give an outline of the main steps of the proof. We set [cf. (18)]

$$T_t{}^\lambda = \sum_{n=0}^{\infty}(i\lambda)^{2n}R_n(t) \tag{35}$$

$R_n(t)$ is a sum over $2^n(2n)!/n!$ terms. We have 2^{2n} terms arising from the commutators. They are indexed by the permutations π as in (19) and by the position δ of A either to the left, $\delta = 1$, or to the right, $\delta = 0$, of t_1. To each π, δ we have $(2n)!/(n!\,2^n)$ pairings p of the integers $\{1,\ldots,2n\}$ arising from the correlation functions $\langle v_{x_{2n}}\cdots v_{x_1}\rangle$ [cf. (12)]. Therefore

$$R_n(t) = \sum_{\pi,\delta,p} R_n^{\pi\delta p}(t) \tag{36}$$

$R_n^{\pi\delta p}(t)$ is the time integral over $R_n^{\pi\delta p}(t_1,\ldots,t_{2n})$:

$$R_n^{\pi\delta p}(t) = \int_{0\leqslant t_1\leqslant\cdots\leqslant t_{2n}\leqslant t} R_n^{\pi\delta p}(t_1,\ldots,t_{2n}) \tag{37}$$

The proof involves three steps.

(i) For $\nu\geqslant 3$ we show the bound

$$|(\phi, R_n(t)A\psi)| \leqslant \|\phi\|\,\|\psi\|\,\|\hat{A}\|_1 C^n t^n \tag{38}$$

If $\tau_0 C<1$, we can therefore interchange the limit as $\lambda\to 0$ and the sum over n.

(ii) We call a pairing p trivial for the permutation π [cf. (19)] if any odd $\pi(j)$ is paired with $\pi(j) + 1$. For $\nu \geqslant 3$ we prove the bound

$$|(\phi, R_n^{\pi\delta p}(t)A\psi)| \leqslant \|\phi\| \, \|\psi\| C' t^{n-\epsilon} \tag{39}$$

$\epsilon > 0$, for all pairings p nontrivial for π. Therefore, if p is nontrivial for π

$$\lim_{\lambda \to 0} (i\lambda)^{2n} R_n^{\pi\delta p}(\lambda^{-2}\tau)A = 0 \tag{40}$$

(iii) If p is trivial for π, we prove the existence of the weak limit

$$\lim_{\lambda \to 0} (i\lambda)^{2n} R_n^{\pi\delta p}(\lambda^{-2}\tau)A = R_n^{\pi\delta}(\tau)A \tag{41}$$

and show that all terms sum up to

$$\sum_{n=0}^{\infty} \sum_{\pi,\delta} R_n^{\pi\delta}(\tau)A = S_{-\tau}^0 S_\tau A \tag{42}$$

Ad (i): We assume $A \in \mathscr{S}(\mathbb{R}^\nu)$. In (20) (with $\Lambda = \Gamma$) we insert the expression (12) for the correlation function. We also insert the explicit form of f following from (15). Then

$$\begin{aligned}
&(\phi, R_n^{\pi\delta p}(t_1, \dots, t_{2n})A\psi) \\
&\quad = \mathrm{si}(\pi)\,(2\pi)^{-n\nu}(\mathrm{BZ})^{-n/2} \\
&\quad \times \sum_{x_1,\dots,x_{2n} \in \Gamma} \int_{\mathrm{BZ}(\Gamma)^n} dk_1 \cdots dk_n \\
&\quad \times \int_{\mathbb{R}^{\nu(2n+1)}} d\Theta_1 \cdots d\Theta_{2n+1}\, \hat{\phi}^*(\Theta_{2n+1}) A(\Theta_{k+\delta}) \hat{\psi}(\Theta_1) \\
&\quad \times \exp[-i(\Theta_1{}^2 t_{\pi(1)} - \Theta_{2n+1}^2 t_{\pi(2n)})] \left\{ \prod_{j=2}^{2n} \exp[-i\Theta_j{}^2(t_{\pi(j)} - t_{\pi(j-1)})] \right\} \\
&\quad \times \exp[i\lambda^2(E\Theta_1 t_{\pi(1)}^2 - E\Theta_{2n+1} t_{\pi(2n)}^2)] \left\{ \prod_{j=2}^{2n} \exp[i\lambda^2 E_j t(\Theta_{\pi(j)}^2 - t_{\pi(j-1)}^2)] \right\} \\
&\quad \times \left\{ \prod_{j=1}^{2n} \hat{V}(\Theta_{j+1} - \Theta_j) \exp[ix_j(\Theta_{j+1} - \Theta_j)] \right\} \\
&\quad \times \left\{ \prod_{j=1}^{n} \hat{g}(k_j) \exp[ik_j(x_{p(2j)} - x_{p(2j-1)})] \right\}
\end{aligned} \tag{43}$$

The sign $\mathrm{si}(\pi)$ of the permutation π is determined by the following rule: If we have the order $\cdots t_{2j} \cdots A \cdots t_{2j-1} \cdots$ or vice versa, we multiply by -1, and otherwise by 1; $j = 1,\ldots, n$. If A is to the left of t_1, then $\delta = 1$; if A is to the right of t_1, then $\delta = 0$. We have

$$(\mathrm{BZ})^{-1} \sum_{x\in\Gamma} e^{ix\Theta} = \sum_{\gamma\in\Gamma^*} \delta(\Theta - \gamma) \tag{44}$$

where γ runs over the reciprocal lattice Γ^*. In (43) we therefore obtain the δ-functions

$$\begin{aligned} &\delta(\Theta_{p(2j-1)+1} - \Theta_{p(2j-1)} - k_j + \gamma_{n+j}), \\ &\delta(\Theta_{p(2j)+1} - \Theta_{p(2j)} + k_j - \gamma_j - \gamma_{n+j}), \qquad j = 1,\ldots, n. \end{aligned} \tag{45}$$

The basic strategy for proving (38) is to exploit the spreading of the wave packet due to the terms $\exp(i\Theta_j{}^2 t_i)$. For this purpose (43) has to be rearranged somewhat. We perform the integration over the Θ variables (except for Θ_1) by using the δ-functions (45). We extend the k integration over all $\mathbb{R}^\nu$, thereby taking care of the summation over $\gamma_{n+1},\ldots, \gamma_{2n}$. We write A as a Fourier transform

$$A(\Theta_{k+\delta}) = \int dy\, e^{iy\Theta_{k+\delta}} \hat{A}(y) \tag{46}$$

By estimating the Θ_1 and the y integrations we obtain finally

$$\begin{aligned} &|(\phi, R_n^{\pi\delta p}(t_1,\ldots, t_{2n})A\psi)| \\ &\quad\leqslant (2\pi)^{-n\nu}(\mathrm{BZ})^{3n/2}\|\phi\|\,\|\psi\|\,\|\hat{A}\|_1 \\ &\quad\times \sup_{\Theta\in\mathbb{R}^\nu} \sup_{y\in\mathbb{R}^\nu} \sum_{\gamma_1,\ldots,\gamma_n\in\Gamma^*} \left| \int_{\mathbb{R}^{\nu n}} dk_1 \cdots dk_n \exp[-iQ^{\pi,p}(k)] \right. \\ &\quad\times \prod_{j=1}^{n} \exp(ik_j l_j) \prod_{j=1}^{n} \hat{V}(k_j)\hat{V}(-k_j + \gamma_j)\hat{g}(k_j) \end{aligned} \tag{47}$$

$Q^{\pi,p}(k)$ is a quadratic form in $k_1,\ldots, k_n$ depending on $t_1,\ldots, t_{2n}$, the permutation π, and the pairing p. The l_j depend on δ, Θ_1, y, $t_1,\ldots, t_{2n}$, t, and $\lambda^2 E$, but do not depend on the k_j. This is a fortunate circumstance, for the spreading of the wave packet is independent of the addition of a form linear in k to $Q^{\pi,p}(k)$. Therefore the specific form of the l_j is of no importance. The form of the $Q^{\pi,p}$ matrix will be essential for the estimates in Lemmas A1 and A2, but does not interest us at the moment. If we regard the last product in (47) as a product of wave functions, we can apply the following result:

The Basic Estimate (Spreading of the Wave Packet). Let Q be a real, symmetric, n-dimensional quadratic form, and let l be a linear functional and $\psi_1, \ldots, \psi_n \in \mathscr{S}(\mathbb{R}^\nu)$. Let P be a projection diagonal in the given representation of Q. By $\det PQP$ we mean the determinant of the principal minor corresponding to P. For $P = 0$ we set $\det PQP = 1$. Then

$$\left| \int_{\mathbb{R}^{\nu n}} \{\exp[i(\langle k|Qk\rangle + \langle l|k\rangle)]\}\psi_1(k_1) \cdots \psi_n(k_n)\, dk_1 \cdots dk_n \right| \leqslant |\det PQP|^{-\nu/2} \left\{ \prod_{j=1}^{n} c_j \right\} \tag{48}$$

where c_j is the maximum of

$$\int_{\mathbb{R}^\nu} |\psi_j(k_j)|\, dk_j \quad \text{and} \quad (4\pi)^{-\nu/2} \int_{\mathbb{R}^\nu} dy\, (1 + y^2)^{-m} \int_{\mathbb{R}^\nu} |(1 - \Delta_j)^m \psi_j(k_j)|\, dk_j \tag{49}$$

with $\nu + 1 \leqslant 2m \leqslant \nu + 2$.

Remark. We need the projection P, since the bound by $|\det Q|^{-\nu/2}$ becomes poor as any of the eigenvalues of Q becomes small. The general bound by $\prod \|\psi_j\|_1$ is not sufficient.

Proof. First we show (48) for $P = 1$. Let λ_j, $j = 1, \ldots, n$, be the eigenvalues of Q and let O diagonalize Q. Viewing $\exp(i\langle k|Qk\rangle)$ as distribution, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{\nu n}} \{\exp[i(\langle k|Qk\rangle + \langle l|k\rangle)]\}\psi_1(k_1) \cdots \psi_n(k_n)\, dk_1 \cdots dk_n \right| \\
&\quad = \prod_{j=1}^{n} |4\pi\lambda_j|^{-\nu/2} \left| \int_{\mathbb{R}^\nu} dx_1 \cdots dx_n \prod_{j=1}^{n} \exp[(i/4\lambda_j)x_j^2] \right. \\
&\qquad \left. \times \int_{\mathbb{R}^{\nu n}} dk_1 \cdots dk_n\, [\exp(i\langle x + Ol|k\rangle)]\psi_1((O^*k)_1) \cdots \psi_n((O^*k)_n) \right| \\
&\quad \leqslant (4\pi)^{-n\nu/2} |\det Q|^{-\nu/2} \int_{\mathbb{R}^{\nu n}} dx_1 \cdots dx_n \\
&\qquad \times \left| \int_{\mathbb{R}^{\nu n}} [\exp(i\langle x|k\rangle)]\psi_1(k_1) \cdots \psi_n(k_n)\, dk_1 \cdots dk_n \right| \\
&\quad \leqslant |\det Q|^{-\nu/2} \left[(4\pi)^{-\nu/2} \int_{\mathbb{R}^\nu} dy\, (1 + y^2)^{-m} \right] \left\{ \prod_{j=1}^{n} \int_{\mathbb{R}^\nu} dk_j\, |(1 - \Delta_j)^m \psi_j(k_j)| \right\}
\end{aligned} \tag{50}$$

If $P \neq 1$, then

Eq. (48)

$$= \int_{\mathbb{R}^{\nu n}} dk_1 \cdots dk_n \exp\{i[\langle k|PQPk\rangle + \langle l|Pk\rangle + 2\langle k|PQ(1-P)k\rangle]\}$$
$$\times \left\{\prod_{j\in P} \psi_j(k_j)\right\} \exp\{i[\langle k|(1-P)Q(1-P)k\rangle + \langle l|(1-P)k\rangle]\}$$
$$\times \left\{\prod_{j\in 1-P} \psi_j(k_j)\right\} \tag{51}$$

and the estimate follows as before. ■

Ad (i) Continued: Applying the basic estimate to (47), we conclude that for all diagonal projections P

$$|(\phi, R_n^{\pi\delta p}(t_1,\dots,t_{2n})A\psi)| \leqslant \|\phi\|\,\|\psi\|\,\|\hat{A}\|_1 C^p |\det PQ^{\pi,p}P|^{-\nu/2} \tag{52}$$

where

$$C = \sum_{\gamma\in\Gamma^*} c(\gamma) \tag{53}$$

with $c(\gamma)$ the maximum of

$$(2\pi)^{-\nu}(\mathrm{BZ})^{3/2}\int_{\mathbb{R}^\nu} dk\,|\hat{V}(k)\hat{V}(-k+\gamma)\hat{g}(k)| \qquad \text{and} \tag{54}$$

$$(2\pi)^{-\nu}(4\pi)^{-\nu/2}(\mathrm{BZ})^{3/2}\int_{\mathbb{R}^\nu} dy\,(1+y^2)^{-m}\int_{\mathbb{R}^\nu} dk\,|(1-\Delta)^m\hat{V}(k)\hat{V}(-k+\gamma)\hat{g}(k)|$$

By the assumption on V and g, C is finite. By Lemma A1 of the appendix

$$|(\phi, R_n^{\pi\delta p}(t)A\psi)| \leqslant \|\psi\|\,\|\phi\|\,\|\hat{A}\|_1(1/n!)(C\|h\|_1)^n t^n \tag{55}$$

We have $2^n(2n)!/n!$ such terms, which implies

$$|(\phi, R_n(t)A\psi)| \leqslant \|\phi\|\,\|\psi\|\,\|\hat{A}\|_1 \frac{2n!\,2^n}{n!\,n!}(C\|h\|_1)^n t^n \tag{56}$$

$(2n)!/(n!\,n!)$ behaves for large n as 4^n. Therefore (56) proves the estimate (38) with $\tau_0 8C\|h\|_1 < 1$.

If we estimate (47) directly (formally, $P = 0$), then we obtain the bound

$$|(\phi, R_n(t)A\psi)| \leqslant \|\phi\|\,\|\psi\|\,\|\hat{A}\|(1/2n!)C^n t^{2n} \tag{57}$$

with C given by (53). This shows that the series (18) for T_t^λ converges in norm for all potentials such that $C < \infty$. A sufficient condition is obviously $g \in l_1(\Gamma)$ and $V, \hat{V} \in L^1(\mathbb{R}^\nu)$.

Ad (ii): By (52) and Lemma A2 of the appendix, for any pairing p nontrivial for π

$$|(\phi, R_n^{\pi\delta p}(t)A\psi)| < \|\phi\|\,\|\psi\|C't^{n-\epsilon} \tag{58}$$

proving (39).

Ad (iii): Let $\phi, \psi \in \mathscr{S}(\mathbb{R}^\nu)$. By (58) in the limit as $\lambda \to 0$ only the pairing trivial for π survives. In the following, p always denotes the trivial for π. We introduce the difference variables

$$s_j = t_{2j} - t_{2j-1}$$

and the rescaled times

$$\tau_j = \lambda^2 t_{2j}$$

($j = 1,\dots, n$). Then

$$(i\lambda)^{2n}(\phi, R_n(\lambda^{-2}\tau)A\psi) = \sum_\pi (-1)^n \int_0^\tau d\tau_n \int_0^{\lambda^{-2}\tau_n} ds_n \int_0^{\tau_n - \lambda^2 s_n} d\tau_{n-1} \cdots \int_0^{\tau_2-\lambda^2 s_2} d\tau_1 \int_0^{\lambda^{-2}\tau_1} ds_1\, (\phi, R_n^{\pi\delta p}(s_1,\dots, s_n, \tau_1,\dots, \tau_n)A\psi) \quad (59)$$

We want to show first that from the sum over $\gamma_1,\dots,\gamma_n$ in (43) with p trivial for π only the term with $\gamma_1 = \cdots = \gamma_n = 0$ survives in the weak coupling limit. For this purpose the k-variables are not so convenient. Therefore, we express in (43), with the help of (45), the k integration in terms of the Θ integration. Then the n δ-functions

$$\delta(\Theta_{p(2j)+1} - \Theta_{p(2j)} + \Theta_{p(2j-1)+1} - \Theta_{p(2j-1)} - \gamma_j)$$

($j = 1,\dots, n$) remain. We perform the integration over the δ-functions in such a way that the integrations over $\Theta = \Theta_1$ and $\Theta_{\omega(j)}$ with $\omega(j) = p(2j-1)+1$, $j = 1,\dots, n$, remain. Let $\bar\omega(j) = p(2j) - 2m$, where $m \geqslant 0$ is the smallest integer such that $\bar\omega(j)$ is one of the $\omega(j)$, and let $\epsilon_j = 1$ if t_{2j} is to the left of t_{2j-1}, and $\epsilon_j = -1$ if t_{2j} is to the right of t_{2j-1}. [The permutation π is ordered from the right: $\pi(2n)\cdots\pi(1)$.] Then the integrand of (59) is given by

$$\begin{aligned}(\phi, R_n^{\pi\delta p}&(s_1,\dots, s_n, \tau_1,\dots, \tau_n)A\psi)\\ &= \mathrm{si}(\pi)\,(2\pi)^{-n\nu}(\mathrm{BZ})^{3n/2}\\ &\times \sum_{\gamma_1,\dots,\gamma_n\in\Gamma^*} \int_{\mathbb{R}^{\nu(n+1)}} d\Theta\, d\Theta_{\omega(1)}\cdots d\Theta_{\omega(n)}\, \hat\phi^*(\Theta + \gamma_1 + \cdots + \gamma_n)A(\Theta_{k+\delta})\\ &\times \hat\psi(\Theta)\left\{\prod_{j=1}^n \hat V(\Theta_{\omega(j)} - \Theta_{\bar\omega(j)})\hat V(-\Theta_{\omega(j)} + \Theta_{\bar\omega(j)} + \gamma_j)\hat g(\Theta_{\omega(j)} - \Theta_{\bar\omega(j)})\right\}\\ &\times \left(\prod_{j=1}^n \exp\{-is_j\epsilon_j[(\Theta_{\omega(j)} - E\tau_j)^2 - (\Theta_{\bar\omega(j)} - E\tau_j)^2]\}\right.\\ &\times \left.\exp[-i\lambda^2\epsilon_j s_j^2 E(\Theta_{\omega(j)} - \Theta_{\bar\omega(j)})]\right) \exp\{i[\tilde Q(\gamma) + l(\gamma)]\}\end{aligned} \quad (60)$$

Either $\delta = 1$ or δ is the largest nonpositive, even integer such that $k + \delta$ is one of the $\omega(j)$. Here $\tilde{Q}$ is a quadratic form and l a linear functional. They arise from terms of the form

$$\begin{aligned}
&\Theta_{p(2j)+1}(t_{\pi(p(2j)+1)} - t_{\pi(p(2j))}) \\
&\quad = \lambda^{-2}(\Theta_{\omega(j)} + \gamma_j + \cdots + \gamma_{j'})^2(\tau_m - \tau_{m'} + \lambda^2 s) \\
&\qquad \times \lambda^2 E\Theta_{p(2j)+1}(t^2_{\pi(p(2j)+1)} - t^2_{\pi(p(2j))}) \\
&\quad = \lambda^{-2}E(\Theta_{\omega(j)} + \gamma_j + \cdots + \gamma_{j'})(\tau_m{}^2 - \tau^2_{m'} + 2\lambda^2 s\tau_m - 2\lambda^2 s\tau_{m'} + s\lambda^4) \qquad (61)
\end{aligned}$$

with $m \neq m'$. Here s stands symbolically for s_m, $s_{m'}$, their difference, or zero. The bound (56) allows us to interchange the limit as $\lambda \to 0$ and the sum over the γ as long as $\tau < \tau_0$. Necessarily, $j < j'$ and $m \neq m'$. Therefore for $(\gamma_1, \ldots, \gamma_n) \neq 0$ we obtain rapidly oscillating terms as λ becomes small, which give zero contribution as integrated over $(\tau_1, \ldots, \tau_n)$. Therefore in the limit as $\lambda \to 0$ only the term with $\gamma_1 = \cdots = \gamma_n = 0$ survives.

To prove the desired limit, we need yet another set of integration variables, after performing the integration over the n δ-function just below (59). Let $\tilde{\pi}$ be a permutation such that for odd $\pi(j)$, $\pi(j) + 1$ is to the left of $\pi(j)$. If $\langle p(2i-1)p(2i)\rangle$ is the pairing associated with t_{2j-1} and t_{2j}, we set $\Theta_j{}' = \Theta_{\omega(j)}$. For given $\tilde{\pi}$ the remaining integration variables are $\Theta = \Theta_1$, $\Theta_j{}'$, $j = 1, \ldots, n$. To each such permutation $\tilde{\pi}$ we associate $2^n - 1$ other permutations with the same sign $\mathrm{si}(\tilde{\pi})$: $\cdots t_{2j}t_{2j-1} \cdots A \cdots$ is associated with $\cdots A \cdots t_{2j-1}t_{2j} \cdots$ and $\cdots t_{2j} \cdots A \cdots t_{2j-1}$ is associated with $\cdots t_{2j-1} \cdots A \cdots t_{2j} \cdots$. In (60) this changes the sign of ϵ_j and allows us therefore to extend the integration over s_j from $-\lambda^{-2}\tau_j$ to $\lambda^{-2}\tau_j$. This introduces in the upper limit of integration of τ_j an error of order $\lambda^2 s_j$, which is negligible in the limit as $\lambda \to 0$. Therefore

$$\begin{aligned}
&\lim_{\lambda\to 0} (-1)^n \sum_{\tilde{\pi},\delta} \mathrm{si}(\tilde{\pi}) \int_0^\tau d\tau_n \cdots \int_0^{\tau_2} dt_1 \int_{-\lambda^{-2}\tau_1}^{\lambda^{-2}\tau_1} ds_1 \int_{\mathbb{R}^{\nu(n+1)}} d\Theta\, d\Theta_1{}' \cdots d\Theta_n{}' \\
&\quad \times (2\pi)^{-n\nu}(\mathrm{BZ})^{3n/2}\hat{\phi}^*(\Theta)A(\Theta'_{\kappa(j)})\hat{\psi}(\Theta) \\
&\quad \times \left(\prod_{j=1}^n \exp\{-is_j[(\Theta_j{}' - E\tau_j)^2 - (\Theta'_{\tilde{\omega}(j)} - E\tau_j)^2]\}|\hat{V}(\Theta_j{}' - \Theta'_{\tilde{\omega}(j)})|^2\right. \\
&\quad \times \left.\hat{g}(\Theta_j{}' - \Theta'_{\tilde{\omega}(j)})\right) \prod_{j=1}^n \exp[\pm i\lambda^2 s_j{}^2 E(\Theta_j{}' - \Theta'_{\tilde{\omega}(j)})] \\
&\quad = (-1)^n \sum_{\tilde{\pi},\delta} \mathrm{si}(\tilde{\pi}) \int_{0\leqslant\tau_1\cdots\leqslant\tau_n\leqslant\tau} d\tau_n \cdots d\tau_1 \int_{\mathbb{R}^{\nu(n+1)}} d\Theta\, d\Theta_1{}' \cdots d\Theta_n{}' \\
&\qquad \times \hat{\phi}^*(\Theta)\hat{\psi}(\Theta)\left\{\prod_{j=1}^n K(\Theta_j{}' - E\tau_j, \Theta'_{\tilde{\omega}(j)} - E\tau_j)\right\}A(\Theta'_{\kappa(j)}) \qquad (62)
\end{aligned}$$

where

$$K(\Theta, \Theta') = (\mathrm{BZ})^{3/2}\, \delta(\Theta^2 - \Theta'^2)|\hat{V}(\Theta - \Theta')|^2 \hat{g}(\Theta - \Theta') \tag{63}$$

$\bar{\omega}(j)$ is determined by the same rule as above, $\kappa(j) = 1$ if $\delta = 1$ for $\tilde{\pi}$ and $\kappa(j) = m$ if $\delta = 0$ and if we have the order $\cdots t_{2m} \cdots t_2 t_1 A t_{2m-1} \cdots$ for $\tilde{\pi}$. We recognize (62) as the Dyson expansion

$$(\phi, S^0_{-\tau} S_\tau \psi) = \int_{0 \leqslant \tau_1 \cdots \leqslant \tau_n \leqslant \tau} d\tau_n \cdots d\tau_1\, (\phi, L(\tau_n) \cdots L(\tau_1) A\psi) \tag{64}$$

where $L(\tau) = S^0_{-\tau} L S_\tau^0$. Since $\|T^\lambda_{\lambda^{-2}\tau} A\| \leqslant \|A\|$, the weak convergence on a dense set in $\mathscr{H}$ implies weak convergence. Since $\mathscr{S}(\mathbb{R}^\nu)$ is norm dense in $C_\infty(\mathbb{R}^\nu)$ and since $\|T_t^\lambda\| \leqslant 1$, $\|S^0_{-t} S_t\| \leqslant 1$, we have weak convergence for all $A \in C_\infty(\mathbb{R}^\nu)$. ■

The proof shows that the requirements of Theorem 2 on the covariance $\langle v_x v_0 \rangle = g_x$ and on the potential V are somewhat too strong. Sufficient conditions are $\hat{g} \in C^{2m}(\mathrm{BZ}(\Gamma))$, $V \in L^1(\mathbb{R}^\nu)$, $\hat{V} \in C^{2m}(\mathbb{R}^\nu)$ such that all derivatives up to order $2m$ are integrable, $\nu + 1 \leqslant 2m \leqslant \nu + 2$.

4. CONCLUSIONS AND COMMENTS

(i) The proof we gave is unsatisfactory in two ways. (1) Certainly the restriction $\tau < \tau_0$ and quite likely the restriction $\nu \geqslant 3$ are artifacts of our method. Physically, one would expect the weak convergence in (34) for all τ and $\nu = 1, 2, \ldots$. (2) Our proof depends in a very delicate way on the fact that the dispersion law is $\omega(k) = k^2$. If ω is only slightly varied, the proof breaks down. The reason is that it is not clear how to extend the basic estimate (with the right n dependence!) to the other dispersions.

(ii) If so wanted, one also can give the Hamiltonian H^λ [cf. (1)] a dynamical (nonrandom) interpretation. v_x is then to be interpreted as the position of the oscillator at site x of an infinitely extended quantum harmonic crystal (which has its own independent dynamics). This means that the ideal solid interacts with the conduction electron by a one-phonon–electron process. The average over the Gaussian measure is replaced by the average over the thermal equilibrium state of the harmonic crystal. We have then the same rules for expressing the higher order correlation functions in terms of the (now time-dependent) two-point function $\langle v_x(t) v_y \rangle$. If we choose the harmonic crystal as a set of independent oscillators all with the same frequency ω, then our estimates still hold. The collision kernel K is given by

$$K(k, k') = |\hat{V}(k - k')|^2 (2\omega)^{-1} [(1 - e^{-\beta\omega})^{-1}\, \delta(k^2 - k'^2 + \omega) + (e^{\beta\omega} - 1)^{-1}\, \delta(k^2 - k'^2 - \omega)] \tag{65}$$

This is the so-called Fröhlich model[6] and describes the interaction of electrons with the optical mode. The weak coupling limit for the polaron with less trivial dynamics for the harmonic crystal is an open problem.

(iii) Davies[7] studied the weak coupling limit for a finitely extended quantum system coupled to a thermal reservoir (an infinite, quasifree Fermi system at equilibrium). It should be noted that in this model the weak coupling limit is controlled in a way very different from ours. Davies has to estimate time integrals over equilibrium time correlation functions of the reservoir. The existence of the weak coupling limit is completely independent of the system Hamiltonian (as long as it has a pure point spectrum). The limit is, so to speak, reservoir-induced. Quite to the contrary, in our case the existence of the weak coupling limit depends strongly on the system Hamiltonian. It is tempting to apply Davies' analysis to the present model. [In order to do this we have to interpret v_x as described in (iii).] If one goes through Davies' proof in a formal way, one ends up with the condition

$$\sum_{x\in\Gamma} |\langle v_0 v_x(t)\rangle| \in L^1(\mathbb{R}) \tag{66}$$

for the equilibrium time correlation function $\langle v_0 v_x(t)\rangle$ of the isolated crystal. Unfortunately, since (66) does not tend to zero as $t \to \infty$, it has no hope of being integrable. It would be of interest to see whether one could modify Davies' method in such a way that the weak coupling limit for the present model becomes controlled through the dynamics of the crystal.

(iv) We also investigated the case of a classical particle moving through random impurities with a Hamiltonian function given by (1) ($E = 0$). In this case one uses a space and time rescaling

$$\lambda \to 0, \qquad \lambda^2 t = \tau \text{ fixed}, \qquad \lambda^2 r = q \text{ fixed} \tag{67}$$

[The rescaling (67) should come as no surprise. Martin[8] has shown that for the classical Lorentz gas (wind-tree model) the limit (67) is equivalent to the Grad limit, where the density of the scatterers is increased and their cross section is decreased in such a way that the mean free path of the wind-particle is kept constant (no space and time rescaling). The equivalence of the two limits should be true in general.] We can copy step by step the proof for the quantum case. In the limit (67) one obtains, again in the interaction picture, the Fokker–Planck type of equation for the time evolution of phase functions $f(p, q)$,

$$\frac{\partial}{\partial t} f(p, q) = \sum_{i,j=1}^{\nu} \frac{\partial}{\partial p_i} h_{ij}(p) \frac{\partial}{\partial p_j} f(p, q) \tag{68}$$

with

$$h_{ij}(p) = \int_0^\infty dt \int_{\mathbb{R}^\nu} dk\, \hat{g}(k)|\hat{V}(k)|^2 k_i k_j e^{ikpt} \tag{69}$$

However, we have been unable to control the interchange of the sum over n and the limit $\lambda \to 0$.

(v) In solid state physics one is interested in a somewhat different kind of random impurity problem. One supposes that the scatterers are randomly distributed in space, e.g., as the spatial part of the ideal gas in equilibrium. For a low density ρ of the random scatterers—a condition which is clearly met in many practical situations—the validity of the transport equation (3) with $\hat{g}(k) = 1$ is assumed. As in Section 2, one can prove for this model the existence of the averaged dynamics in the infinite-volume limit and the convergence of the Dyson expansion. In the limit density $\rho \to 0$, $t \to \infty$ such that $\rho t = \tau$ is kept constant (there is no assumption on the weakness of the interaction!), one obtains formally the transport equation (3). The mechanism for the survival of those terms that add up to the transport equation is precisely the same as in the present model. Unfortunately, in the Dyson expansion at each order n there are so many terms that a uniform estimate in τ, even for short times, cannot be obtained.

APPENDIX

Let p be a pairing of $\{1,..., 2n\}$ and let π be a permutation of $\{1,..., 2n\}$ such that $\pi(j+1) > \pi(j)$ for $j \geqslant k$ and $\pi(j+1) < \pi(j)$ for $k \geqslant j+1$. We order $\{1,..., 2n\}$ and the permutation π from the right, i.e., $2n \cdots j \cdots 1$ and $\pi(2n) \cdots \pi(j) \cdots \pi(1)$. The pairings are labeled in order of their first member to the right, i.e., $p(1) < \cdots < p(2j-1) < \cdots < p(2n-1)$. Let

$$s_j = t_{\pi(j+1)} - t_{\pi(j)}, \qquad j = 1,..., 2n-1 \tag{A1}$$

We conclude from (45) that the $n \times n$ matrix $Q^{\pi,p}$ in (46) is given by

$$(Q^{\pi,p})_{jm} = \sum_i s_i \tag{A2}$$

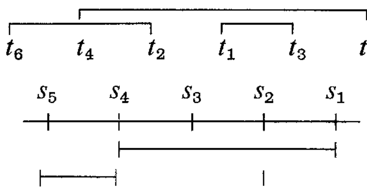
where the sum is over all i such that $p(2j-1) \leqslant i \leqslant p(2j) - 1$ and $p(2m-1) \leqslant i \leqslant p(2m) - 1$. Let h be the function

$$h: t \to \min\{1, |t|^{-\nu/2}\} \tag{A3}$$

Let P be a projection diagonal in the given representation of $Q^{\pi,p}$. By $\det PQ^{\pi,p}P$ we mean the determinant of the principal minor corresponding to P. For $P = 0$ we set $\det\ PQ^{\pi,p}P = 1$.

The analysis leading to the results of Lemmas A1 and A2 is rather involved. However, the basic strategy is very simple and can be easily understood in terms of a graphical representation of $Q^{\pi,p}$, which we introduce now. At the beginning of each proof we will explain then the "flow diagram" in terms of this graphical representation.

We fix the permutation π and introduce the difference variables s_j. For example [here $\pi(1) = 5$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 2$, $\pi(5) = 4$, $\pi(6) = 6$, $p(1) = 5$, $p(2) = 4$, $p(3) = 3$, $p(4) = 1$, $p(5) = 2$, $p(6) = 6$]



(*)

The specific pairing p determines the diagonal elements of $Q^{\pi,p}$, which are represented as straight lines, as can be seen from the example. The off-diagonal elements of $Q^{\pi,p}$ are already uniquely determined by its diagonal elements. (The off-diagonal term jm contains all terms common to the diagonal terms jj and mm.) For example, (*) is the graphical representation of

$$Q^{\pi,p} = \begin{pmatrix} s_1 + s_2 + s_3 + s_4 & s_2 & s_4 \\ s_2 & s_2 & 0 \\ s_4 & 0 & s_4 + s_5 \end{pmatrix}$$

We have the following bound:

Lemma A1.

$$\int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_{2n} \leqslant t} dt_{2n} \cdots dt_1 \min_P |\det PQ^{\pi,p}P|^{-\nu/2} \leqslant (1/n!) t^n (\|h\|_1)^n \tag{A4}$$

for $\nu \geqslant 3$.

Proof. (0) We express the time integration in terms of the s-variables [step (i)]. One has to perform $2n$ time integrations. For the integration over the s-variables appearing at the first place from the right in a line one uses an L^1-norm estimate of $\min_P |\det PQ^{\pi,p}P|^{-\nu/2}$. One has to show [step (ii)] that after this estimate the remaining function comes from a new $Q^{\pi,p}$ matrix that is the old one where the line used in the estimate has been simply omitted. Since we have n lines, we obtain thereby the factor $(\|h\|_1)^n$. The remaining n integrations are performed explicitly. At each step this results in a polynomial. If there is a norm estimate, one takes the supremum of the polynomial over the range of integration. One has to show [step (iii)] that the explicit integrations and the suprema together result in $(1/n!)t^n$. (In our example, we estimate the s_1 and the s_2 integrations, perform explicitly the s_3 integration, estimate the s_4 integration and take the supremum of the polynomial, and integrate explicitly over s_5 and t_6.)

(i) To perform the integration, we express the integral (A4) in terms of the s-variables, which will depend then on the permutation π. Let $l(j)$ be the largest integer such that $\pi(l(j)) < \pi(j)$. Then

$$s_j + \cdots + s_m \begin{cases} \leqslant 0 & \text{if} \quad m < l(j) \\ \geqslant 0 & \text{if} \quad m \geqslant l(j) \end{cases} \tag{A5}$$

$(m = j,..., 2n - 1)$. We assume that $\pi(2n) = 2n$. The case $\pi(1) = 2n$ can be obtained by reflection. Then

$$0 \leqslant s_j + \cdots + s_{2n-1} \leqslant t_{2n} \tag{A6}$$

From (A5) and (A6) we obtain the domain of integration

$$-(s_{j+1} + \cdots + s_{l(j)}) \leqslant s_j \leqslant 0 \tag{A7a}$$

if $j > k$ and $\pi(l(j)) < \pi(j) - 1$;

$$-(s_{j+1} + \cdots + s_{l(j)}) \leqslant s_j \leqslant -(s_{j+1} + \cdots + s_{l(j)-1}) \tag{A7b}$$

if $j < k$ and $\pi(l(j)) = \pi(j) - 1$; and

$$0 \leqslant s_j \leqslant t_{2n} - (s_{j+1} + \cdots + s_{2n-1}) \tag{A7c}$$

if $j \geqslant k$.

(ii) We exploit the special structure of $Q^{\pi,p}$. Since

$$p(1) = 1 \tag{A8}$$

s_1 enters only at $(Q^{\pi,p})_{11}$. Therefore

$$\det Q^{\pi,p} = s_1(\det Q^{\pi;p}_{2,n}) + R_1 \tag{A9}$$

where R_1 is a rest independent of s_1. The index $2, n$ denotes the principal minor from 2 to n. (It can be shown that $\det Q^{\pi,p}$ is a monomial with coefficients $+1$ in the s-variables.) $Q^{\pi;p}_{2;n}$ has the same structure as $Q^{\pi;p}_{1;n}$ but depends only on $s_{p(3)},..., s_{2n-1}$. As we integrate over $s_1,..., s_{r-1}$ the integrand has the following form:

$$\min_P |\det PQ^{\pi;p}_{j-n}P|^{-\nu/2} \operatorname{pol}(s_r,..., s_{2n-1}, t_{2n})(\|h\|_1)^m \tag{A10}$$

where pol denotes some polynomial. Two cases can now occur:

1. The first term does not depend on s_r. Then the integration over s_r will result in a new polynomial $\operatorname{pol}'(s_{r+1},..., s_{2n-1}, t_{2n})$ with a degree raised by one.

2. The first term depends on s_r. Then $r = p(2j-1)$ and we estimate by

$$\sup_{s_{p(2j-1)}\in[a,b]} |\mathrm{pol}(s_{p(2j-1)},\ldots, s_{2n-1}, t_{2n})|(\|h\|_1)^m \times \int_a^b ds_{p(2j-1)} \min_P |\det PQ_{j,n}^{\pi,p}P|^{-\nu/2} \tag{A11}$$

Let the unit vector in direction j be contained in the range of P. Then we have

$$|\det PQ_{j,n}^{\pi,p}P|^{-\nu/2} = |\det PQ_{j+1,n}^{\pi,p}P|^{-\nu/2}|s_{p(2j-1)} + R_j(\det PQ_{j+1,n}^{\pi,p}P)^{-1}|^{-\nu/2} \tag{A12}$$

where R_j is some rest independent of $s_{p(2j-1)}$. The integrand in (A11) is also bounded by $|\det PQ_{j+1,n}^{\pi,p}P|^{-\nu/2}$. Therefore

$$(\mathrm{A11}) \leqslant \sup_{s_{p(2j-1)}\in[a,b]} |\mathrm{pol}(s_{p(2j-1)},\ldots, s_{2n-1}, t_{2n})|(\|h\|_1)^{m+1} \times \min_P|\det PQ_{j+1,n}^{\pi,p}P|^{-\nu/2} \tag{A13}$$

and we pick up an extra power of $\|h\|_1$ and a polynomial of the same degree as before.

(iii) We have to keep track of the polynomials. Let $\pi(i) < \cdots < \pi(j)$ be all those permutations having the same $l(j)$. If we integrate the constant function over $s_j,\ldots, s_i$, we obtain

$$(1/\alpha_j!)(s_{l(j)})^{\alpha_j} \tag{A14}$$

with $\alpha_j = \pi(l(j)+1) - \pi(l(j)) - 1$. If we now integrate over

$$\min_P|\det PQ^{\pi,p}P|^{-\nu/2}$$

then we have to apply the bounds developed in (ii). In each case the sup of the polynomial is at the upper limit of integration. Consequently, at each estimate we pick up one power of $\|h\|_1$ and one power less of the polynomial we obtained before. Therefore, depending on the number of estimates, the integration now results in

$$(1/\alpha_j'!)(s_{l(j)})^{\alpha_j'}(\|h\|_1)^{m'} \tag{A15}$$

with $\alpha_j' = \pi(l(j)+1) - \pi(j) - 1 - m'$. After all integrations up to s_{k-1}, the integrand is then given by

$$\left\{\prod_{j=k}^{2n-1} \frac{1}{\alpha_j!}(s_j)^{\alpha_j}\right\}(\|h\|_1)^m \min_P|\det PQ_{m+1,n}^{\pi,p}P|^{-\nu/2} \tag{A16}$$

where $0 \leqslant \alpha_j \leqslant \pi(j+1) - \pi(j) - 1$ and

$$\sum_{j=k}^{2n-1} \alpha_j + m = \sum_{j=k}^{2n-1} (\pi(j+1) - \pi(j) - 1)$$

The first $\alpha_j \neq 0$ is for $j = l(k-1)$. Let $\pi(l(k-1)) - \pi(k) = \beta'$. Integrating over $s_k, \ldots, s_{l(k-1)-1}$ gives the factor

$$(1/\beta'!)(t_{2n} - s_{2n-1} - \cdots - s_{l(k-1)})^{\beta'}$$

If we have to apply the estimate (A13) several times, then the integrand becomes

$$\left\{\prod_{j=l(k-1)+1}^{2n-1} \frac{1}{\alpha_j!}(s_j)^{\alpha_j}\right\} \frac{1}{\alpha_{l(k-1)}!}(s_{l(k-1)})^{\alpha_{l(k-1)}}$$
$$\times \frac{1}{\beta!}(t_{2n} - s_{2n-1} - \cdots - s_{l(k-1)})^{\beta}(\|h\|_1)^{m'} \min_P |\det PQ^{\pi,p}_{m'+1,n}P|^{-\nu/2} \quad \text{(A17)}$$

If $\det Q^{\pi,p}_{m'+1,n}$ does not depend on $s_{l(k-1)}$, then we obtain upon integration the factor

$$\left\{\prod_{j=l(k-1)+1}^{2n-1} \frac{1}{\alpha_j!}(s_j)^{\alpha_j}\right\} \frac{1}{(\alpha_{l(k-1)} + \beta + 1)!}(t_{2n} - s_{2n-1} - \cdots - s_{l(k-1)+1})^{(\alpha_{l(k-1)}+\beta+1)} \quad \text{(A18)}$$

If $\det Q^{\pi,p}_{m'+1,n}$ depends on $s_{l(k-1)}$, we apply (A13) and estimate the supremum on (A17). We obtain

$$\left\{\prod_{j=l(k-1)+1}^{2n-1} \frac{1}{\alpha_j!}(s_j)^{\alpha_j}\right\} \frac{1}{(\alpha_{l(k-1)} + \beta)!}(t_{2n} - s_{2n-1} - \cdots - s_{l(k-1)+1})^{(\alpha_{l(k-1)}+\beta)}$$
$$\times (\|h\|_1)^{m'+1} \min_P |\det PQ^{\pi,p}_{m'+2,n}P|^{-\nu/2} \quad \text{(A19)}$$

where we used

$$\log \alpha! + \log \beta! + (\alpha + \beta)\log(\alpha + \beta) - \alpha \log \alpha - \beta \log \beta \geqslant \log(\alpha + \beta)!$$

Continuing the integration, we find

$$\int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_{2n} \leqslant t} dt_{2n} \cdots dt_1 \min_P |\det PQ^{\pi,p}P|^{-\nu/2}$$
$$\leqslant \int_0^t dt_{2n} \frac{1}{(n-1)!}(t_{2n})^{n-1}(\|h\|_1)^n = \frac{1}{n!} t^n (\|h\|_1)^n \quad \blacksquare \quad \text{(A20)}$$

We call a pairing p trivial for π if any odd $\pi(j)$ is paired with $\pi(j) + 1$.

Lemma A2. If p is nontrivial for π and $\nu \geqslant 3$, then

$$\int_{0 \leqslant t_1 \cdots \leqslant t_{2n} \leqslant t} dt_{2n} \cdots dt_1 \min_P |\det PQ^{\pi,p}P|^{-\nu/2} \leqslant Ct^{n-\epsilon} \quad \text{(A21)}$$

for some $\epsilon > 0$.

Proof. We restrict ourselves again to the case $\pi(2n) = 2n$. We isolate an important condition for the pairings p, which will be used throughout the proof:

$$\begin{aligned}
&1.\ p(2j) - 1 = p(2j-1) \quad \text{if} \quad p(2j-1) \geqslant k \\
&2.\ \text{either } p(2j) - 1 = p(2j-1) \quad \text{or} \quad p(2j) - 1 = l(p(2j-1)) \\
&\quad \text{if} \quad p(2j-1) < k \quad \text{and} \quad \pi(l(p(2j-1))) < \pi(p(2j-1)) - 1 \qquad (*) \\
&3.\ \text{either } p(2j) - 1 = l(p(2j-1)) \\
&\quad\ \text{or} \quad p(2j) - 1 = l(p(2j-1)) - 1 \\
&\quad \text{if} \quad (2j-1) < k \quad \text{and} \quad \pi(l(p(2j-1))) = \pi(p(2j-1)) - 1
\end{aligned}$$

(0) We fix the permutation π and consider the set of all pairings. In step (i) we will exploit the fact that

$$\int_y^\infty dx\, h(x), \quad y > 0 \qquad \text{and} \qquad \int_{-\infty}^y dx\, h(x), \qquad y < 0$$

behave as $|y|^{-\epsilon}$, $\epsilon > 0$, for large $|y|$, which gives after all remaining integrations $t^{n-\epsilon}$. We perform all integrations until there is only one line left. (Up to s_3 in our example.) If this line is not of the form given in (*), then all terms with these pairings will behave as $t^{n-\epsilon}$. We therefore consider now only the subset of all pairings for which the last line of the graphical representation has the form (*). We perform all integrations up to the line before the last one. Again, if this line is not of the form (*), then all terms with these pairings will behave as $t^{n-\epsilon}$. We therefore consider now only the subset of all pairings for which the last line and the line before the last one has the form (*), etc. The general case from j to $j + 1$ is treated in step (ii). (In our example the last line $s_4 - s_5$ gives already the $-\epsilon$ power.)

In step (iii) we consider the subset of all pairings for which (*) is satisfied. Except for the trivial pairing, each one of these terms necessarily contains an integral of the form

$$\int_0^y dx\, x^q h(x) \sim y^{q-\epsilon}$$

($y > 0$, $q \geqslant 1$, $\epsilon > 0$), which upon integration leads to $t^{n-\epsilon}$. We conclude therefore that for a given permutation π all terms except the one with the trivial pairing behave as $t^{n-\epsilon}$.

(i) Let us assume that all integrations up to $s_{p(2n-1)-1}$ have been performed as in Lemma A1. Then, up to a polynomial, the integrand is

$$h(s_{p(2n-1)} + \cdots + s_{p(2n)-1})$$

Integrating over $s_{p(2n-1)}$, we pick up the power of $-\epsilon$ except if conditions (*) are satisfied for $j = n$.

To see this, we write out the $s_{p(2n-1)}$ integration and perform a change of variables:

$$\begin{aligned}
&1_+. \quad \int_{s_{p(2n-1)+1}+\cdots+s_{p(2n)-1}}^{t_{2n}-(s_{p(2n)}+\cdots+s_{2n-1})} dx\, h(x) \\
&2_+. \quad \int_{s_{l(p(2n-1))+1}+\cdots+s_{p(2n)-1}}^{s_{p(2n-1)+1}+\cdots+s_{p(2n)-1}} dx\, h(x) \\
&2_-. \quad \int_{-(s_{p(2n)}+\cdots+s_{l(p(2n-1))})}^{s_{p(2n-1)+1}+\cdots+s_{p(2n)-1}} dx\, h(x) \qquad \text{(A22)} \\
&3_+. \quad \int_{s_{l(p(2n-1))+1}+\cdots+s_{p(2n)-1}}^{s_{l(p(2n-1))}+\cdots s_{p(2n)-1}} dx\, h(x) \\
&3_-. \quad \int_{-(s_{p(2n)}+\cdots+s_{l(p(2n-1))})}^{-(s_{p(2n)}+\cdots+s_{l(p(2n-1))-1})} dx\, h(x)
\end{aligned}$$

The respective conditions are

1_+. $p(2n-1) \geqslant k$
2_+. $p(2n-1) < k$ and $p(2n)-1 > l(p(2n-1))$
2_-. $p(2n-1) < k$ and $p(2n)-1 < l(p(2n-1))$

with $\pi(l(p(2n-1))) < \pi(p(2n-1)) - 1$; and conditions 3_+ and 3_- as conditions 2_+ and 2_- with $\pi(l(p(2n-1))) = \pi(p(2n-1)) - 1$. Let

$$f_+(y) = \int_y^\infty dx\, h(x), \qquad f_-(y) = \int_{-\infty}^y dx\, h(x) \qquad \text{(A23)}$$

Then f_+ behaves as $y^{-\epsilon}$ for large, positive values and f_- as $|y|^{-\epsilon}$ for large, negative values. The integrals indicated by the plus sign can be estimated by f_+ and those indicated by the minus sign by f_-, which gives the extra power $-\epsilon$.

(ii) Let us assume that all integrations up to $s_{p(2m-1)-1}$ have been performed and assume that in the integrations over $s_{p(2j-1)}$, $j = m+1,\ldots,n$, we will pick up the extra power $-\epsilon$ except if for $j = m+1,\ldots,n$ together the conditions (*) are satisfied. We want to show that we pick up the power $-\epsilon$ except if for m the conditions (*) are also satisfied.

We have to consider only those pairings for which (*) are satisfied for $j = m+1,\ldots,n$. The possible pairings $\langle p(2m-1)p(2m)\rangle$ are classified as (a) nonoverlapping if $p(2m) < p(2m+1)$, (b) partially overlapping if

$p(2j-1) \leqslant p(2m) - 1 < p(2j) - 1$ for some $j = m+1, \ldots, n$, (c) completely overlapping otherwise.

If the pairing $\langle p(2m-1)p(2m)\rangle$ is nonoverlapping, then by the same method as in (i), we pick up the power $-\epsilon$ except if (*) are satisfied with $j = m$.

If the pairing $\langle p(2m-1)p(2m)\rangle$ is completely overlapping, then we distinguish again the five cases above. We find now integrals of the form

$$\int dx_1\, f_+(a - x_1)h(x_1), \qquad \int dx_1\, dx_2\, f_+(a - x_1 - x_2)h(x_1)h(x_2), \ldots$$
$$\int dx_1\, f_-(-a - x_1)h(x_1), \qquad \int dx_1\, dx_2\, f_-(-a - x_1 - x_2)h(x_1)h(x_2), \ldots \tag{A24}$$

with $a > 0$, which behave for large a as $a^{-\epsilon}$. We denote these cases briefly by $f_+(a - x)h(x)$ and $f_-(-x - a)h(x)$. We assume that (*) is not satisfied for $j = m$. Then for 1_+ we find $f_+(a)$. For 2_+ and 3_+ we obtain either $f_+(a)$ or $f_+(a - x)h(x)$. For 3_- we obtain $f_-(-a - x)h(x)$. The final case 2_- is somewhat more complicated. We estimate at the upper limit of the integral as $f_-((s_{p(2m-1)+1} + \cdots + s_{p(2m-1)}) - (\cdots))$. Let m' be the largest integer with $(m') = l(p(2m-1))$. By (*) and the complete overlapping for $p(2j-1) < m'$ we can have only "points," i.e., factors of the type $h(s_{p(2j-1)})$, $j > m$. If this already exhausts all pairings $j = m+1, \ldots, n$, then, by consecutively taking the supremum of f_-, we obtain $f_-(-a)$. If this does not exhaust all other pairings, then by estimating the supremum of f_- we obtain

$$f_-(-(s_{p(2m)} + \cdots + s_{l(p(2m-1))-1}) - (\cdots))$$

which is of the form $f_-(-a - x)h(x)$.

If the pairing $\langle p(2m-1)p(2m)\rangle$ is partially overlapping, then of the five cases (A22), only 2_- and 3_- can occur. We note that for large s

$$\int_{-\infty}^{0} dy \min\{1, |y + s|^{-\nu/2}, |y|^{-\nu/2+1}\} \sim |s|^{-\epsilon} \tag{A25}$$

If $s \leqslant 0$, the claim is obvious. If $s \geqslant 0$, we choose around the singularity $y = -s$ the interval $[-s - s^q, -s + s^q]$ with $0 < q < \frac{1}{3}$. Then (A25) can be split as

$$\int_{-\infty}^{-s-s^q} dy\, |y + s|^{-\nu/2} + \int_{-s-s^q}^{-s+s^q} dy\, |y|^{-\nu/2+1} + \int_{-s+s^q}^{\infty} dy\, |y + s|^{-\nu/2} \tag{A26}$$

The first and the last terms behave as $s^{-\nu q/2}$ and the middle term as $s^{-(\nu/2)+1+q}$.

To estimate the partially overlapping pairings, let $\langle p(2j-1)p(2j)\rangle$ be the first pairing (from the right) after $\langle p(2m-1)p(2m)\rangle$ such that $p(2j) -$

$1 > p(2j-1)$. We estimate the $s_{p(2m-1)}$ integration and all integrations between $s_{p(2m-1)}$ and $s_{p(2j-1)}$ as in Lemma A1. Then the remaining determinant factorizes and we obtain for the $s_{p(2j-1)}$ integration the bound

$$\min\{1, |s_{p(2j-1)} + s'|^{-\nu/2}\} \tag{A27}$$

where s' stands for a sum of the variables $s_{p(2j-1)+1},\ldots, s_{p(2j-1)}$ depending on p. Furthermore, we obtain the bound

$$\int_{-\infty}^{0} ds_{p(2m-1)} \, |s_{p(2m-1)} + \cdots + s_{p(2m)-1}|^{-\nu/2}$$
$$\leqslant c|s_{p(2j-1)} + \cdots + s_{p(2m-1)-1}|^{-\nu/2+1} \tag{A28}$$

since $s_{p(2j-1)} + \cdots + s_{p(2m)-1} \leqslant 0$. If $s_{p(2j-1)+1} + \cdots + s_{p(2m)-1} \geqslant 0$, then we can bound (A28) by $|s_{p(2j-1)}|^{-\nu/2+1}$. Then (A25) gives the desired extra power of $-\epsilon$. If $s_{p(2j-1)+1} + \cdots + s_{p(2m)-1} \leqslant 0$, then the upper limit of integration for $s_{p(2j-1)}$ is necessarily $-(s_{p(2j-1)+1} + \cdots + s_{l(p(2j-1))-1})$. Shifting the integration limits by $s_{p(2j-1)+1} + \cdots + s_{p(2m)-1}$, we can again apply (A25) with $s = s' - (s_{p(2j-1)+1} + \cdots + s_{p(2m)-1})$ to obtain the power $-\epsilon$.

(iii) By (ii) only the pairings satisfying (*) for $j = 1,\ldots, n$ do not give rise to the extra power $-\epsilon$. Here we want to show that among all pairings satisfying (*) for $j = 1,\ldots, n$ only the trivial pairing does not pick up a power of $-\epsilon$. We note that because of (*), $\det PQ^{\pi,p}P$ factorizes.

We look at the pairing joining $\pi(k) - 1$ and $\pi(m)$. [To simplify the notation we consider here p as a pairing of $\pi(2n) \cdots \pi(1)$ and not of $2n \cdots 1$ as before.] If $m > k$, then by (*), $m = k + 1$ and $\pi(m)$ is even, say $2b$. Since to the right of k we necessarily have then the pairings $\langle 2\,3\rangle,\ldots, \langle 2b-2\;2b-1\rangle$, for the integration over s_k we obtain $(s_k)^\alpha$, where $\alpha \geqslant 1$ whenever $b > 1$. The integral

$$\int_0^{t_{2n}-(s_{k+1}+\cdots+s_{2n-1})} ds_k \, (s_k)^\alpha h(s_k) \sim [t_{2n} - (s_{k+1} + \cdots + s_{2n-1})]^{\alpha-\epsilon} \tag{A29}$$

for $\alpha \geqslant 1$. If $k > m$, then $\pi(m) = 2$. Therefore only the pairing $\langle 1\,2\rangle$ does not pick up an extra power of $-\epsilon$.

Let us suppose that we showed already that only the pairings $\langle 1\,2\rangle,\ldots, \langle 2d-1\;2d\rangle$ do not have the extra power $-\epsilon$. We look at the pairing joining $\pi(j) = 2d + 1$ and $\pi(m)$. Since by assumption $\pi(j) < \pi(m)$, by (*), $\pi(j) + 1 = \pi(m)$ whenever $\pi(j)$ is second (from the right) in the pairing. Let us suppose that $\pi(j)$ is first in the pairing. If $j > k$, we can repeat the argument of the preceding paragraph to conclude that only for $\pi(j) + 1 = \pi(m)$ is there no extra power of $-\epsilon$. If $j < k$, then by (*) necessarily $m > k$, $\pi(m)$ even, say

$2b$, and to the right of j we have the pairings $\langle \pi(j) + 2 \rangle, \ldots, \langle 2b - 2\, 2b - 1 \rangle$. Therefore we obtain the integral

$$\int_{-(s_{j+1} + \cdots + s_{l(j)})}^{0} ds_j \, (s_j + \cdots + s_{l(j)})^{\alpha} h(s_j + \cdots' + s_{l(j)}) \tag{A30}$$

where $\alpha \geqslant 1$ whenever $\pi(j) + 1 < \pi(m)$. The prime indicates that we do not have all variables in between. The upper limit of the integral may be also $-(s_{j+1} + \cdots + s_{l(j)-1})$. If $\alpha \geqslant 1$, the integral (A30) gives the extra power of $-\epsilon$. Therefore only for $\pi(j) + 1 = \pi(m)$ do we not obtain the extra power of $-\epsilon$. This concludes the proof. ■

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